Limit Theorems

Definition (c.f. Definition 3.2.1)**.** A sequence (x_n) is said to be *bounded* if there exists a real number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Exercise. Prove that (x_n) is bounded if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded.

Theorem (c.f. Theorem 3.2.2)**.** *A convergent sequence of real numbers is bounded.*

Example 1. Observe the following examples:

- The sequence $(1/n)$ is convergent, **hence** it is bounded by this theorem.
- The sequence $((-1)^n)$ is bounded but not convergent.
- The sequence (n) is unbounded, **hence** it is not convergent.
- There is no convergent sequence that is unbounded.

Theorem (c.f. Theorem 3.2.3)**.** *Let (x_n) and (y_n) be sequences of real numbers that converges to x and y respectively, and let $c \in \mathbb{R}$.*

(a) The sequence $(x_n \pm y_n)$ converges to $x \pm y$.

(b) The sequence (cx_n) converges to cx.

(c) If $y_n \neq 0$ for all $n \in \mathbb{N}$ and $y \neq 0$, the sequence (x_n/y_n) converges to x/y.

Remark. Before applying the theorem, make sure that the sequences are convergent.

Example 2 (c.f. Section 3.2, Ex.6)**.** Find the limits of the following sequences:

(a) $\lim\limits_{n\to\infty} \left(2 + \dfrac{1}{n}\right)^2$ (b) $\lim\limits_{n\to\infty} \dfrac{3n-8}{2n+9}$

Solution. We avoid guessing the limits and then prove it using "ε-N" arguments.

(a) Using the fact that $\lim b = b$ and $\lim(1/n) = 0$, where $b \in \mathbb{R}$ is a constant,

$$\begin{aligned}\lim_{n\to\infty} \left(2 + \frac{1}{n}\right)^2 &= \lim_{n\to\infty} \left(4 + \frac{4}{n} + \frac{1}{n^2}\right) \\ &= \lim_{n\to\infty} 4 + 4 \cdot \left(\lim_{n\to\infty} \frac{1}{n}\right) + \left(\lim_{n\to\infty} \frac{1}{n}\right) \cdot \left(\lim_{n\to\infty} \frac{1}{n}\right) \\ &= 4 + 4 \cdot 0 + 0 \cdot 0 \\ &= 4.\end{aligned}$$

(b) Using the fact that $\lim(1/n) = 0$,

$$\lim_{n\to\infty} \frac{3n-8}{2n+9} = \lim_{n\to\infty} \frac{3 - 8/n}{2 + 9/n} = \frac{3-0}{2+0} = \frac{3}{2}.$$

Exercise. Evaluate $\lim\limits_{n\to\infty} \left(\dfrac{2}{3n} - 7\right)^3$ and $\lim\limits_{n\to\infty} \dfrac{14n - 38}{6n^2 - 7n + 9}$.

Theorem (c.f. Theorem 3.2.5). *Let* (x_n) *and* (y_n) *be convergent sequences satisfying*

$$x_n \le y_n, \quad \forall n \in \mathbb{N}\,.$$

Then $\lim(x_n) \le \lim(y_n)$*.*

Squeeze Theorem (c.f. 3.2.7). *Let* (x_n)*,* (y_n) *and* (z_n) *be sequences. Suppose that*

$$x_n \le y_n \le z_n, \quad \forall n \in \mathbb{N}.$$

and $\lim(x_n) = \lim(z_n)$*. Then* (y_n) *is convergent and* $\lim(x_n) = \lim(y_n) = \lim(z_n)$*.*

Remark. Notice the following:

- In the **assumption** of each theorem, the sign "$\le$" can be replaced by "$<$". However, we cannot do so for the statements in the **conclusion**.
- Since only the tails of the sequences affect the limit behaviour, the condition in the assumption can be relaxed to hold for all $n \ge K$, where K is some natural number.
- Notice that (y_n) is not assumed to be convergent in the **Squeeze Theorem**. Once its convergence is established, the result $\lim(x_n) = \lim(y_n) = \lim(z_n)$ is immediate.

Example 3. Observe the following examples:

- Note that $1/n > 0$ for all $n \in \mathbb{N}$. Hence $\lim(1/n) \ge 0$. ($\lim(1/n) > 0$ is **false**.)
- Note that $1/(n - 10.5) \ge 0$ for all $n \ge 11$. Hence $\lim(1/n - 10.5) \ge 0$.
- We have $\lim(\sin n/n) = 0$. To see this, note that $-1 \le \sin n \le 1$ for all n and hence

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Example 4 (c.f. Section 3.2, Ex.15). Show that if $z_n = (a^n + b^n)^{1/n}$ where $0 < a < b$, then

$$\lim(z_n) = b.$$

Solution. We want to apply the **Squeeze Theorem**. Since $0 < a < b$,

$$b^n < a^n + b^n < 2b^n, \quad \forall n \in \mathbb{N}.$$

Therefore $b < z_n < b \cdot 2^{1/n}$ for all $n \in \mathbb{N}$. Together with the fact that

$$\lim(b) = b \quad \text{and} \quad \lim(b \cdot 2^{1/n}) = b \cdot \lim(2^{1/n}) = b.$$

Hence by **Squeeze Theorem**, (z_n) is convergent and $\lim(z_n) = b$.

Monotone Sequences

Definition (c.f. Definition 3.3.1)**.** Let (x_n) be a sequence of real numbers.

- (x_n) is said to be *increasing* if $x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} \le \cdots$.
- (x_n) is said to be *decreasing* if $x_1 \ge x_2 \ge \cdots \ge x_n \ge x_{n+1} \ge \cdots$.
- (x_n) is said to be *monotone* if it is either increasing or decreasing.

Monotone Convergence Theorem (c.f. 3.3.2)**.** *A monotone sequence (x_n) of real numbers is convergent if and only if it is bounded. Moreover,*

- *if (x_n) is increasing, then $\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$.*
- *if (x_n) is decreasing, then $\lim(x_n) = \inf\{x_n : n \in \mathbb{N}\}$.*

Example 5 (c.f. Section 3.3, Ex.1)**.** Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Hence find its limit.

Solution. Let calculate the first few terms to get a feeling about this sequence:

$$(x_n) = (8, 6, 5, 4.5, 4.25, ...)$$

To show that this sequence is bounded, we show $4 \le x_n \le 8$ for all $n \in \mathbb{N}$ by induction. The case $n = 1$ is obvious. Now suppose $4 \le x_n \le 8$ for some n, then

$$\frac{1}{2}(4) + 2 \le \frac{1}{2}x_n + 2 \le \frac{1}{2}(8) + 2.$$

i.e., $4 \le x_{n+1} \le 6 \le 8$. We then show that $x_n - x_{n+1} \ge 0$ for all n, so that (x_n) decreasing:

$$x_n - x_{n+1} = x_n - \left(\frac{1}{2}x_n + 2\right) = \frac{1}{2}x_n - 2 \ge \frac{1}{2}(4) - 2 = 0$$

Since this sequence is bounded and monotone, it is convergent by **Monotone Convergence Theorem**. Let $x = \lim(x_n)$ be its limit. Passing the limit into the inductive formula

$$x = \frac{1}{2}x + 2,$$

it follows that the required limit is $x = 2$.

Let (x_n) be a **bounded** sequence of real numbers. For each $n \in \mathbb{N}$, define the set

$$X_n = \{x_k : k \ge n\} = \{x_n, x_{n+1}, x_{n+2}, ...\}.$$

Notice that $X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supseteq X_{n+1} \supseteq \cdots$. Hence the subset relation yields

$$\inf X_1 \le \inf X_2 \le \cdots \le \inf X_n \le \cdots \le \sup X_n \le \cdots \le \sup X_2 \le \sup X_1.$$

By **Monotone Convergence Theorem**, we have the following definition:

Definition (c.f. Definition 3.4.10 and Theorem 3.4.11)**.** Let (x_n) be a **bounded** sequence.

- The *limit superior* of (x_n), denoted by $\limsup(x_n)$ or $\overline{\lim}(x_n)$, is given by

$$\overline{\lim}(x_n) = \inf_n \left(\sup_{k \geq n} x_k \right) = \lim_{n\to\infty} \left(\sup_{k\geq n} x_k \right).$$

- The *limit inferior* of (x_n), denoted by $\liminf(x_n)$ or $\underline{\lim}(x_n)$, is given by

$$\underline{\lim}(x_n) = \sup_n \left(\inf_{k \geq n} x_k \right) = \lim_{n\to\infty} \left(\inf_{k\geq n} x_k \right).$$

Remark. The definitions of limit superior and limit inferior in the textbook are complicated. The above definition is an equivalent formulation given in Theorem 3.4.11.

Theorem (c.f. Theorem 3.4.12 and Section 3.4, Ex.18)**.** *A bounded sequence (x_n) is convergent if and only if $\underline{\lim}(x_n) = \overline{\lim}(x_n)$.*

Exercise. Prove the above theorem.

Example 6. Determine whether the sequence $\big((-1)^n/n\big)$ is convergent or not.

Solution. Let $x_n = (-1)^n/n$. We calculate the limit inferior and limit superior of (x_n):

$$\underline{\lim}(x_n) = \sup_n \left[\inf_{k\geq n} \left(\frac{(-1)^k}{k} \right) \right] = \sup_n \left(\frac{1}{n'} \right) = 0$$
$$\overline{\lim}(x_n) = \inf_n \left[\sup_{k\geq n} \left(\frac{(-1)^k}{k} \right) \right] = \inf_n \left(\frac{1}{n''} \right) = 0$$

Here, n' and n'' denote the smallest even and odd integer not less than n respectively. Since $\underline{\lim}(x_n) = \overline{\lim}(x_n) = 0$, the sequence converges to 0.

Example 7. Determine whether the sequence $\big(\sin(n\pi/4)\big)$ is convergent or not.

Solution. Let $y_n = \sin(n\pi/4)$. We calculate the limit inferior and limit superior of (y_n):

$$\underline{\lim}(y_n) = \sup_n \left[\inf_{k\geq n} \left(\sin\frac{n\pi}{4} \right) \right] = \sup_n \left(\inf \left\{ 0, \pm\frac{\sqrt{2}}{2}, \pm 1 \right\} \right) = -1$$
$$\overline{\lim}(y_n) = \inf_n \left[\sup_{k\geq n} \left(\sin\frac{n\pi}{4} \right) \right] = \inf_n \left(\sup \left\{ 0, \pm\frac{\sqrt{2}}{2}, \pm 1 \right\} \right) = 1$$

Since $\underline{\lim}(y_n) \neq \overline{\lim}(y_n)$, the sequence does not converge.

Exercise. Determine whether $\big((-1)^n/n^2\big)$ and $\big(\cos(n\pi/3)\big)$ are convergent or not.